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New integrable sectors in the Skyrme and four-dimensional CP^n models

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Abstract

The application of a weak integrability concept to the Skyrme and CP^n models in four dimensions is investigated. A new integrable subsystem of the Skyrme model, allowing also for non-holomorphic configurations, is derived. This procedure can be applied to the massive Skyrme model as well. Moreover, an example of a family of chiral Lagrangians providing exact, finite energy Skyrme-like solitons with arbitrary value of the topological charge is given. In the case of CP^n models a tower of integrable subsystems is obtained. In particular, in (2+1) dimensions a one-to-one correspondence between the standard integrable submodel and the BPS sector is proved. Additionally, it is shown that weak integrable submodels allow also for non-BPS solutions. Geometric as well as algebraic interpretations of the integrability conditions are also given.

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1. Introduction

Among nonlinear field theories, many models with topological solitons appear to possess particular importance in various branches of physics (see for example [1–3]). In (1+1) dimensions one can take advantage of the well-known methods to study the mathematical structure and dynamics of such objects (called kinks or alternatively domain walls), such as the inverse scattering method, Lax pair formalism or the Bäcklund transformation [4, 5]. In principle, these methods are connected with the integrability property of models, which implies the existence of infinitely many conserved currents. As a consequence, the chances for the construction of exact, topological solutions are strongly correlated with the integrability. Unfortunately, in higher spatial dimensions there are no such general tools allowing for a systematical construction of solitons. What one has instead are integrable subsectors, such as the holomorphic solitons of Belavin and Polyakov in three dimensions, predecessors of the self-dual instantons of Yang–Mills in four dimensions.

However, there is a new promising approach, based on the construction of local zero curvature representations for nonlinear models, which enables one to calculate sufficient conditions for the existence of integrable subsystems, even though the full model does not have necessarily such a property (which would be thereby exhibited if present [6, 7]). Here, integrability is understood as possessing a zero curvature formulation and the existence of an infinite number of conserved currents. The significance of this method originates in the expectation that also in higher dimensional theories integrability is an important step in the way of deriving analytical solutions.

That this works has been shown in investigations of especially useful models with Hopf solitons, where an integrable sector of the Nicole model has been found [8]. It was also proved that the simplest Hopf soliton belongs to this sector. Further, a model with infinitely many exact hopfions has been given [9, 10].

As is well known, a general sufficient condition (the so-called generalized BPS) for the Skyrme model [11–13] analogous to the Baby Skyrme or self-dual cases has not been found. Simplified models based on the Skyrme idea of avoiding the scaling instabilities, like those mentioned above, have been investigated. Among them, the interesting Skyrme–Faddeev–Niemi model. Unfortunately the integrable sector of this well-known model [14, 15] seems to contain no topological solitons. In order to cope with this problem, one natural option is to relax the pertinent integrability condition (which in this case is the eikonal equation [16, 17]). The construction of such weaker integrability conditions has been recently described in [18]. As one might anticipate, the space of solutions of this new, weaker integrable submodel is considerably larger than the standard one. However, the problem whether it is populated by hopfions is still an open question.

In the case of the Skyrme model the zero curvature method also allowed us to discover a special integrable subsystem [19]. The skyrmion with unit topological charge is a solution of this restricted submodel, but solitons with higher value of the baryon number do not belong to the integrable submodel.

The main aim of the present paper is to define, by means of weaker integrability conditions, a new ('weaker') integrable sector of the Skyrme theory with a substantially richer set of configurations, which enhances the chance to find skyrmions among them. This is carried out in section 2, where, in addition, we generalize the procedure of Aratyn, Ferreira and Zimerman and introduce a Skyrme-like model which possesses infinitely many exact, chiral solitons with arbitrary topological charge. Using this toy model one can study in an analytical way chiral solitons, their energies and profiles. It is an alternative approach to the so-called Baby Skyrme models [20, 21] which also allows for analytical treatment of the solitons (baby skyrmions) but in lower dimensional spacetime.

In section 3, we present the investigations of the CP^n model in any dimension with similar results. In particular, in the case of the CP^n model in two spatial dimensions, we discuss connections between strong/weak integrability and BPS/non-BPS sectors. The conclusions are summarized in section 4.

2. Skyrme model

2.1. Standard integrable submodel

The Skyrme model, without potential term for the chiral field, is given by the following formula:

$$L = \frac{f_\pi^2}{4}\,\mathrm{Tr}(U^\dagger\partial_\mu U U^\dagger\partial^\mu U) - \frac{1}{32e^2}\,\mathrm{Tr}[U^\dagger\partial_\mu U, U^\dagger\partial_\mu U]^2, \tag{1}$$

where f_π, e are constants and U is a $SU(2)$-valued matrix field parameterized in the standard manner as

$$U = \mathrm{e}^{\mathrm{i}\xi_i\tau^i}. \tag{2}$$

Here, τ_i, $i = 1, 2, 3$ are the Pauli matrices and ξ_i are real fields. However, using results of [19] it is convenient to take advantage of the slightly different parametrization

$$U = \mathrm{e}^{\mathrm{i}\xi T}. \tag{3}$$

Here

$$\xi \equiv \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \tag{4}$$

and

$$T \equiv \frac{1}{1+|u|^2}\begin{pmatrix} |u|^2 - 1 & -2\mathrm{i}u \\ 2\mathrm{i}u^* & 1 - |u|^2 \end{pmatrix}, \tag{5}$$

where the complex field appears due to the standard stereographic projection

$$\frac{\vec{\xi}}{\xi} = \frac{1}{1+|u|^2}(u + u^*, -\mathrm{i}(u - u^*), |u|^2 - 1). \tag{6}$$

Then, in the notation of [19] we derive the equations of motion

$$D^\mu B_\mu = \partial^\mu B_\mu + [A^\mu, B_\mu] = 0, \tag{7}$$

where

$$A_\mu \equiv \frac{1}{1+|u|^2}\left(-\mathrm{i}\partial_\mu u\tau_+ - \mathrm{i}\partial_\mu u^*\tau_- + \frac{1}{2}(u\partial_\mu u^* - u^*\partial_\mu u)\tau_3\right) \tag{8}$$

$$B_\mu \equiv -\mathrm{i}R_\mu\tau_3 + \frac{2\sin\xi}{1+|u|^2}(\mathrm{e}^{\mathrm{i}\xi}S_\mu\tau_+ - \mathrm{e}^{-\mathrm{i}\xi}S_\mu^*\tau_-), \tag{9}$$

whereas

$$R_\mu \equiv \partial_\mu\xi - 8\lambda\frac{\sin^2\xi}{(1+|u|^2)^2}(N_\mu + N_\mu^*) \tag{10}$$

$$S_\mu \equiv \partial_\mu u + 4\lambda\left(M_\mu - \frac{2\sin^2\xi}{(1+|u|^2)^2}K_\mu\right) \tag{11}$$

and

$$K_\mu \equiv (\partial^\nu u\partial_\nu u^*)\partial_\mu u - (\partial_\nu u)^2\partial_\mu u^* \tag{12}$$

$$M_\mu \equiv (\partial^\nu u\partial_\nu\xi)\partial_\mu\xi - (\partial_\nu\xi)^2\partial_\mu u \tag{13}$$

$$N_\mu \equiv (\partial^\nu u\partial_\nu u^*)\partial_\mu\xi - (\partial_\nu\xi\partial^\nu u)\partial_\mu u^*. \tag{14}$$

In addition, $\tau_\pm \equiv (\tau_1 \pm \mathrm{i}\tau_2)/2$.

It has been proved [19] that the standard integrable sector of the Skyrme model is defined by imposing two constraints

$$S_\mu\partial^\mu u = 0, \qquad R_\mu\partial^\mu u = 0 \tag{15}$$

or in a more restricted form

$$(\partial_\mu u)^2 = 0, \qquad \partial^\mu \xi \partial_\mu u = 0. \tag{16}$$

One can note that the first integrability condition is nothing else but the complex eikonal equation.

Then, one can construct two classes of infinitely many currents

$$J_\mu^G = \frac{\partial G}{\partial u} S_\mu - \frac{\partial G}{\partial u^*} S_\mu^* \tag{17}$$

$$J_\mu^{(H_1,H_2)} = 4 \sin\xi \cos\xi (H_1 S_\mu + H_2 S_\mu^*) - (1+|u|^2)^2 \left(\frac{\partial H_1}{\partial u^*} + \frac{\partial H_2}{\partial u} \right) R_\mu, \tag{18}$$

where G is an arbitrary function of ξ, u, u^* whereas H_1, H_2 depend only on u and u^*. Indeed, such currents are conserved if the fields satisfy the integrability condition (15). The importance of this integrable submodel becomes transparent if one notes that the skyrmions with charges $Q = \pm 1$ belong to it. Also the rational map Ansatz, which is widely used to approximate solitons numerically [22–25] (but which does *not* provide exact soliton solutions for charges $|Q| > 1$), obeys the conditions (15).

On the other hand, it is known that the eikonal equation constrains the space of solutions in a rather considerable way. For example, as was mentioned before, there has been found only one topological soliton in the integrable submodel of the Nicole model. In the Faddeev–Skyrme–Niemi model the integrable submodel seems to contain no solitons.

In the case of the Skyrme model the assumption of the eikonal equation as a constraint results in a restriction of the form of the u field. Concretely, for the separation of variable ansatz $\xi = \xi(r)$ and $u = u(\theta, \phi)$ a u field obeying the eikonal equation must be a (anti)holomorphic function of the variable $z = \tan\left(\frac{\theta}{2}\right) e^{i\phi}$. If we combine it with the requirement of the finiteness of the topological charge, then we find that u must be just a rational map in z and, as a consequence, skyrmions with higher charges do not belong to this integrable sector. Because of that, it is reasonable to seek other integrable submodels, which are defined by weaker integrability conditions.

2.2. *New integrable submodel*

It has been recently demonstrated [18] that for all nonlinear theories in Minkowski space with two-dimensional target space there exists an integrability condition which is weaker than the eikonal equation. Here, we will apply this result in the context of the Skyrme model. This weak integrable model is certainly not empty, because it contains the strong integrable submodel, which has the $Q = \pm 1$ skyrmions as solutions.

We begin with the only assumption that

$$\partial_\mu \xi \partial^\mu u = 0 \tag{19}$$

whereas, contrary to the standard integrability subsector, $(\partial u)^2$ is allowed to take arbitrary, in particular non-zero, values. Thus, we get only the second constraint in equation (15)

$$R_\mu \partial^\mu u = 0. \tag{20}$$

The pertinent equations of motion read

$$\partial_\mu S^\mu = \frac{2}{1+|u|^2} u^* S^\mu \partial_\mu u \tag{21}$$

and

$$\partial_\mu R^\mu = 4 \frac{\sin\xi \cos\xi}{(1+|u|^2)^2} S^\mu \partial_\mu u^*. \tag{22}$$

Since the eikonal equation is not imposed, $S^\mu \partial_\mu u \neq 0$ and the left-hand side in (21) is not identically zero. One can find that

$$S^\mu \partial_\mu u = (\partial u)^2 (1 - 4\lambda^2 (\partial \xi)^2). \tag{23}$$

The first class of conserved currents can be taken as follows:

$$J_\mu^G = G_u S_\mu - G_{u^*} S_\mu^*. \tag{24}$$

In spite of the fact that the form of the currents is identical to (17) there is, however, a subtle modification. Now, an arbitrary function G is assumed to depend on the square of the modulus of u, i.e., $G = G(uu^*, \xi)$.

Then, the divergence reads

$$\begin{aligned}\partial^\mu J_\mu = {} & G_{uu} \partial^\mu u S_\mu + G_{uu^*} \partial^\mu u^* S_\mu + G_u \partial^\mu S_\mu - G_{u^* u^*} \partial^\mu u^* S_\mu^* \\ & - G_{u^* u} \partial^\mu u S_\mu^* - G_{u^*} \partial^\mu S_\mu^* + G_u \partial^\mu \xi S_\mu - G_{u^*} \partial^\mu \xi S_\mu^*.\end{aligned} \tag{25}$$

Taking into account (20) and observing that

$$S^\mu \partial_\mu \xi = 0, \qquad \partial^\mu u^* S_\mu = \partial^\mu u S_\mu^*$$

we get

$$\partial^\mu J_\mu = G'' (u^{*2} \partial^\mu u S_\mu - u^2 \partial^\mu u^* S_\mu^*) + G' (u^* \partial^\mu S_\mu - u \partial^\mu S_\mu^*), \tag{26}$$

where prime denotes differentiation with respect to the modulus squared. Using (21) and (23) one can rewrite it as

$$\begin{aligned}\partial^\mu J_\mu = {} & G'' (1 - 4\lambda^2 (\partial \xi)^2) [u^{*2} (\partial u)^2 - u^2 (\partial u^*)^2] \\ & + G' \frac{2}{1 + |u|^2} (1 - 4\lambda^2 (\partial \xi)^2) [u^{*2} (\partial u)^2 - u^2 (\partial u^*)^2].\end{aligned} \tag{27}$$

Thus, the current is conserved if we assume the weak integrability condition

$$[u^{*2} (\partial u)^2 - u^2 (\partial u^*)^2] = 0. \tag{28}$$

Similarly, one can consider the second type of currents, i.e.,

$$J_\mu^{(H_1, H_2)} = 4 \sin \xi \cos \xi (H_1 S_\mu + H_2 S_\mu^*) - (1 + |u|^2)^2 \left(\frac{\partial H_1}{\partial u^*} + \frac{\partial H_2}{\partial u} \right) R_\mu. \tag{29}$$

It can be easily shown that its divergence vanishes if the functions H_1 and H_2 are of the form

$$H_1 = \frac{\partial h}{\partial u}, \qquad H_2 = -\frac{\partial h}{\partial u^*},$$

where h is any function of the modulus squared uu^*. This strongly simplifies the currents. Namely, they are reduced to the first class of currents J^G.

Let us now summarize the results obtained so far. We have defined a new integrable sector of the Skyrme model which consists of two integrability constraints

$$\partial_\mu \xi \partial^\mu u = 0, \qquad u^{*2} (\partial u)^2 - u^2 (\partial u^*)^2 = 0 \tag{30}$$

and two dynamical equations (21), (22). The pertinent infinite family of conserved currents is parameterized by any function G, which depends on uu^* and ξ in an arbitrary manner

$$J_\mu^G = G_u S_\mu - G_{u^*} S_\mu^*. \tag{31}$$

The geometric meaning of the integrability conditions (30) becomes especially well visible if we express the complex field in terms of the scalars Σ, Λ

$$u = e^{\Sigma + i\Lambda}. \tag{32}$$

Then, using equations (30) we get

$$\partial_\mu \Lambda \partial^\mu \Sigma = 0, \qquad \partial_\mu \Lambda \partial^\mu \xi = 0, \qquad \partial_\mu \Sigma \partial^\mu \xi = 0. \tag{33}$$

In other words, in the integrable sector all gradients of the scalar fields must be mutually perpendicular.

It is worth noting that such an integrable submodel can be constructed for a massive generalization of the Skyrme theory [26] as well. For example, introducing the most typical massive term

$$V = f_\pi^2 m^2 \,\mathrm{Tr}(1 - U) \tag{34}$$

results in a modification of equation (22)

$$\partial_\mu R^\mu = 4\frac{\sin\xi \cos\xi}{(1+|u|^2)^2} S^\mu \partial_\mu u^* + 2m^2 \sin\xi. \tag{35}$$

One can check that it does not influence the definition of the integrable sector and the conserved currents. This may be of some interest since the Skyrme model with a massive term serves considerably better as a model of the low energy QCD, describing low energy degrees of freedom i.e. hadrons and nucleons with acceptable accuracy [27, 28].

As was discussed in the previous subsection, all rational maps (or generally holomorphic functions) obey the standard integrability condition. Obviously, they satisfy the weaker condition as well. However, this weaker condition allows for a much larger class of configurations. For instance, non-holomorphic functions in the form (32) fulfil the constraint. This may be of some interest since it has been recently shown that a non-holomorphic Ansatz approximates true Skyrme solitons considerably better than the standard rational Ansatz [29, 30]. In particular, the non-holomorphic approximation of the skyrmion with topological charge $Q = 2$ [29] (which is *not* an exact solution of the field equations) obeys our integrability condition.

2.3. Integrable chiral model

One might ask whether it is possible to further relax the integrability conditions (30). In fact, one can construct a chiral $SU(2)$ model which is integrable even if the second constraint in (30) is neglected. In other words, there are no additional requirements for the complex field u. This property, as it will be shown below, allows for the existence of solitons with an arbitrary value of the topological charge.

The model we are going to focus on is given by the following Lagrangian density:

$$L = -f(\xi) g(u, u^*) [K_\mu \partial^\mu u^*]^{\frac{3}{4}} + [(\partial_\mu \xi)^2]^{\frac{3}{2}}, \tag{36}$$

where f and g are arbitrary, differentiable functions depending on ξ and u, u^* respectively. Additionally

$$K_\mu = (\partial_\nu u \partial^\nu u^*) \partial_\mu u - (\partial_\nu u \partial^\nu u) \partial_\mu u^*. \tag{37}$$

The fractional exponent presented in the upper formula is understood as follows:

$$[K_\mu \partial^\mu u^*]^{\frac{3}{4}} \equiv K_\mu \partial^\mu u^* |K_\mu \partial^\mu u^*|^{-\frac{1}{4}}. \tag{38}$$

It can be observed that such a form of the model guarantees also an interesting way of circumventing Derrick's theorem since the energy is invariant under scaling transformations. This attempt was originally proposed 30 years ago by Deser *et al* [31] to consider the theory of pion fields π^i and further developed by many authors (see the Nicole model [8], Aratyn–Ferreira–Zimerman model [10] and their generalizations [32, 33]). Let us note that our model can be treated as the modified AFZ model coupled in a non-minimal way (via the dielectric function f) with a non-standard scalar field[3].

In principle, one may rewrite this model in terms of the original chiral field U but in practice such a Lagrangian would take a completely illegible form.

The pertinent equation of motion for the complex scalar field reads

$$\tfrac{3}{2}\partial_\mu[fg[K_\mu\partial^\mu u^*]^{-\frac{1}{4}}K^\mu] - g_{u^*}f[K_\mu\partial^\mu u^*]^{\frac{3}{4}} = 0, \tag{39}$$

where $g_{u^*} \equiv \partial_{u^*}g$. This equation can be simplified to the following expression:

$$\partial_\mu[fg^{\frac{1}{3}}[K_\mu\partial^\mu u^*]^{-\frac{1}{4}}K^\mu] = 0 \tag{40}$$

or in the most compact form

$$\partial_\mu\mathcal{K}^\mu = 0, \tag{41}$$

where

$$\mathcal{K}^\mu = fg^{\frac{1}{3}}[K_\mu\partial^\mu u^*]^{-\frac{1}{4}}K^\mu. \tag{42}$$

The second independent equation of motion, for the real scalar field ξ, is

$$3\partial_\mu[[(\partial_\mu\xi)^2]^{\frac{1}{2}}\partial^\mu\xi] + f_\xi g[K_\mu\partial^\mu u^*]^{\frac{3}{4}} = 0. \tag{43}$$

In order to establish the integrability property of the model we introduce the following currents:

$$J_\mu = \mathcal{K}_\mu G_u - \mathcal{K}^*_\mu G_{u^*}, \tag{44}$$

where G is an arbitrary function of u, u^* and ξ. Taking into account the equations of motion it can be proved that these currents are conserved if we assume only one constraint (19). It is due to the fact that the expression $\mathcal{K}_\mu\partial^\mu u = 0$ is a mathematical identity which does not restrict the form of the complex field.

In order to find topologically non-trivial solutions of these dynamical equations we will consider only time-independent configurations and assume the Ansatz for the fields

$$u = u(\phi, \theta) = \mathrm{e}^{\mathrm{i}n\phi}h(\theta) \tag{45}$$

and

$$\xi = \xi(r), \tag{46}$$

where the spherical coordinates (r, θ, ϕ) have been introduced. Here n is an integer number. Of course, such an Ansatz obeys the integrability condition. Thus, the obtained solitons will belong to the integrable submodel.

One can easily find that

$$\vec{K}\cdot\nabla u^* = 4\left(\frac{nhh_\theta}{\sin\theta r^2}\right)^2 \tag{47}$$

and as a consequence

$$\vec{\mathcal{K}} = \sqrt{2}g^{1/3}f\left(\frac{nhh_\theta}{\sin\theta}\right)^{\frac{1}{2}}\frac{\mathrm{e}^{\mathrm{i}n\phi}}{r^2}\left[0, \frac{nh}{\sin\theta}, \mathrm{i}h_\theta\right]. \tag{48}$$

Substituting these formulae into the pertinent field equation we get the following second order, ordinary differential equation for the function h:

$$\partial_\theta\left[nhg^{1/3}\left(\frac{nhh_\theta}{\sin\theta}\right)^{\frac{1}{2}}\right] - nh_\theta g^{1/3}(h)\left(\frac{nhh_\theta}{\sin\theta}\right)^{\frac{1}{2}} = 0. \tag{49}$$

[3] Models with a non-canonical kinetic term for the scalar field have been recently widely discussed in the cosmology. They have been applied in the context of inflation [34], dark matter [35] as well as the modified Newtonian dynamics (MOND) [36]. For the detailed studies of the kinetic term occurring in our model (36) see [37].

It can be simplified to this expression

$$\partial_\theta \left[g^{1/3} \left(\frac{h h_\theta}{\sin\theta} \right)^{\frac{1}{2}} \right] = 0, \tag{50}$$

which possesses the obvious solution

$$g^{1/3} \left(\frac{h h_\theta}{\sin\theta} \right)^{\frac{1}{2}} = \mu, \tag{51}$$

where μ is a positive constant.

Taking into account the Ansatz and the upper obtained solution (51) we are able to rewrite the equation of motion for ξ in the form

$$\frac{3}{r^2}\partial_r[r^2|\xi_r|\xi_r] - f_\xi \frac{(2n)^{3/2}\mu^3}{r^3} = 0. \tag{52}$$

After introducing a new variable

$$x = \ln r \tag{53}$$

we derive the following equation:

$$\partial_x\, [|\xi_x|\xi_x] - f_\xi \frac{(2n)^{3/2}\mu^3}{3} = 0. \tag{54}$$

Fortunately it can be integrated for any function f. The solution is

$$(\xi_x)^3 = \frac{(2n)^{3/2}}{2}\mu^3 f(\xi). \tag{55}$$

Both formulae (51) and (55) can be integrated, at least using standard numerical methods, for all reasonable functions g and f leading to the following general solutions:

$$\int g^{2/3} h \,\mathrm{d}h = -\mu^2 \cos\theta + \alpha_0 \tag{56}$$

and

$$\int \frac{\mathrm{d}\xi}{\sqrt[3]{f}} = \frac{(2n)^{1/2}}{\sqrt[3]{2}}\mu \ln\frac{r}{r_0}, \tag{57}$$

where α_0, r_0 are integration constants.

Let us now analyse in detail a particular case with the following forms of the coupling functions:

$$g(u, u^*) = \frac{1}{(1+|u|^2)^3} \tag{58}$$

and

$$f(\xi) = \sin^3\xi. \tag{59}$$

The corresponding Lagrangian density can be expressed as follows:

$$L = -\sin^3\xi[\vec{n}\cdot(\partial_\mu\vec{n}\times\partial_\nu\vec{n})]^{\frac{3}{4}} + [(\partial_\mu\xi)^2]^{\frac{3}{2}}, \tag{60}$$

where $\vec{n} = \vec{\xi}/\xi$ is a three-component unit vector.

Then

$$\frac{h h_\theta}{(1+h^2)^2} = \mu^2 \sin\theta. \tag{61}$$

Hence, the solution is given as follows:

$$h^2 = \frac{(1+\alpha_0+\mu^2)\sin^2\frac{\theta}{2} + (1+\alpha_0-\mu^2)\cos^2\frac{\theta}{2}}{(-\alpha_0-\mu^2)\sin^2\frac{\theta}{2} + (-\alpha_0+\mu^2)\cos^2\frac{\theta}{2}}. \tag{62}$$

This solution reduces to a well-known form if the parameters satisfy

$$1+\alpha_0-\mu^2 = 0, \qquad \mu^2+\alpha_0 = 0. \tag{63}$$

That is

$$\alpha_0 = -\tfrac{1}{2}, \qquad \mu^2 = \tfrac{1}{2}. \tag{64}$$

Indeed, then we get

$$h = \tan\frac{\theta}{2}. \tag{65}$$

In other words, we obtained the following complex scalar field:

$$u(\theta,\phi) = \mathrm{e}^{\mathrm{i}n\phi}\tan\frac{\theta}{2}. \tag{66}$$

One can observe that if we take into account the stereographic projection then we find

$$\vec{n} = (\cos n\phi\sin\theta, \sin n\phi\sin\theta, \cos\theta)\,. \tag{67}$$

The remaining field equation

$$\xi_x = \frac{\sqrt{n}}{\sqrt[3]{2}}\sin\xi \tag{68}$$

can be solved as well, providing the solution

$$\xi = 2\arctan\left(\frac{r}{r_0}\right)^{\frac{\sqrt{n}}{\sqrt[3]{2}}}. \tag{69}$$

The topological charge of the obtained configuration can be computed from the standard expression

$$Q = \frac{1}{4\pi^2}\int 2\mathrm{i}\frac{\mathrm{d}u\wedge\mathrm{d}u^*\wedge\mathrm{d}\xi}{(1+|u|^2)^2}. \tag{70}$$

Inserting our solutions (66), (69) into (70) one gets

$$Q = n\int\frac{\mathrm{d}\xi}{\pi} = n\frac{\xi(0)-\xi(\infty)}{\pi} = -n. \tag{71}$$

Therefore the solution represents a chiral field with arbitrary value of the baryon number.

Let us now compute the corresponding total energy. In the case of static configurations it is given by

$$E = \int (f(\xi)g(u,u^*)[\vec{K}\nabla u^*]^{\frac{3}{4}} + [(\nabla\xi)^2]^{\frac{3}{2}})\,\mathrm{d}^3x. \tag{72}$$

In particular, for the example considered above, it can be rewritten as

$$E = \int\left[\left(\frac{n}{2}\right)^{\frac{3}{2}}\frac{\sin^3\xi}{r^3} + (\xi_r')^3\right]r^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r. \tag{73}$$

Taking into account equation (51) we get

$$E = 4\pi\cdot 3\int_0^\infty \mathrm{d}r\, r^2\xi_r^3. \tag{74}$$

Finally, using solution (69) one finds that

$$E = \frac{3\pi^2}{\sqrt[3]{2}} n. \tag{75}$$

As one could expect the energy of the obtained configurations is proportional to the topological charge.

Remark. It is easy to observe that solutions of the model (36) with other coupling function $g(u, u^*)$ have the same topological properties. Of course, this is true only if the pertinent boundary conditions for h and ξ are fulfilled. Namely,

$$\xi \longrightarrow \begin{cases} 0 \text{ or } \pi & \text{if } r \to 0 \\ \pi \text{ or } 0 & \text{if } r \to \infty, \end{cases} \qquad h \longrightarrow \begin{cases} 0 \text{ or } \infty & \text{if } \theta \to 0 \\ \infty \text{ or } 0 & \text{if } \theta \to \infty. \end{cases} \tag{76}$$

For instance, if one considers the function g in the following form:

$$g(u, u^*) = \frac{1}{(1 + |u|^2)^p}, \tag{77}$$

where $p > 3/2$, then the general solution reads

$$h^2 = \left(\frac{3}{2(2p-3)(\mu^2 \cos\theta - \alpha_0)} \right)^{\frac{3}{2p-3}} - 1. \tag{78}$$

The topologically interesting configuration is obtained if the constants are

$$\mu^2 = \frac{3 \cdot 2^{\frac{3-2p}{3}}}{4(2p-3)}, \qquad \alpha_0 = -\mu^2. \tag{79}$$

Indeed,

$$h^2 = \left(\frac{2}{\cos\theta + 1} \right)^{\frac{3}{2p-3}} - 1 \tag{80}$$

is a function which satisfies the required boundary conditions (76). The energy-charge relation remains unchanged. Assuming other forms of the function g, it is possible to construct more complicated but still topologically non-trivial solutions.

Remark. In order to derive solutions with fractional topological charge one can consider the following coupling function:

$$f(\xi) = \sin^3\left(\frac{\xi}{q}\right), \tag{81}$$

where q is a positive parameter. The function g is assumed in the standard form (58) providing the previously described complex field (66). However, the real field ξ is no longer a map onto the segment $[0, \pi]$. Indeed, now it is given by the expression

$$\xi = 2q \arctan\left(\frac{r}{r_0}\right)^{\frac{\sqrt{n}}{q\sqrt[3]{2}}} \tag{82}$$

i.e., $\xi \in [0, q\pi]$. Therefore, the original chiral field U does not cover the whole target space providing an arbitrary, in general non-integer, value of the pertinent topological index

$$Q = -nq. \tag{83}$$

Nonetheless, the energy is still finite

$$E = 3\sqrt[3]{2}\pi^2 nq. \tag{84}$$

As one sees, the energy is proportional to Q even if it takes non-integer value.

The existence of finite energy solutions with any non-integer value of the topological index can indicate that the obtained chiral solitons are unstable. This is in accordance with [37] where we have shown that the static Hopf solitons in the Aratyn–Ferreira–Zimerman model have analogous stability problems.

3. CP^n model

3.1. Standard integrable submodel

Let us now investigate the CP^n model in an arbitrary dimensional spacetime [7, 38–40]

$$L = \frac{(1+\mathbf{u}^\dagger \cdot \mathbf{u})(\partial_\mu \mathbf{u}^\dagger \cdot \partial^\mu \mathbf{u}) - (\mathbf{u}^\dagger \cdot \partial_\mu \mathbf{u})(\partial^\mu \mathbf{u}^\dagger \cdot \mathbf{u})}{(1+\mathbf{u}^\dagger \cdot \mathbf{u})^2}, \tag{85}$$

where $\mathbf{u}$ is a column of n complex fields u_a and $\mathbf{u}^\dagger \cdot \mathbf{u} \equiv \sum_{a=1}^n u_a^* u_a$, $a = 1, \ldots, n$. Let us point out that solitons (i.e., static solutions with finite energy) may exist in this model only for $d = 2$ space dimensions due to Derrick's theorem. However, the integrability considerations which follow are equally valid for time-dependent solutions, so still may be relevant for CP^n models in more than 2+1 dimensions, as well.

The equations of motions for the CP^n model read

$$(1+\mathbf{u}^\dagger \cdot \mathbf{u})\partial^2 u_a = 2(\mathbf{u}^\dagger \cdot \partial_\mu \mathbf{u})\partial^\mu u_a. \tag{86}$$

As it was established in [7] this model also possesses an integrable sector, given by the dynamical equations

$$\partial^2 u_a = 0, \qquad a = 1, \ldots, n \tag{87}$$

and the set of integrability conditions

$$\partial_\mu u_a \partial^\mu u_b = 0, \qquad 1 \leqslant a \leqslant b \leqslant n. \tag{88}$$

In fact, if we assume (87), (88) the following currents are conserved (*no* summation over a)

$$J_\mu^{(a)} = \frac{\partial \mathcal{F}}{\partial u_a}\partial_\mu u_a - \frac{\partial \mathcal{F}}{\partial u_a^*}\partial_\mu u_a^*, \tag{89}$$

where $\mathcal{F}$ is an arbitrary function of $u_1, u_1^* \ldots u_n, u_n^*$. Applying the decomposition of the complex field used before

$$u_a = \mathrm{e}^{\Sigma_a + i\Lambda_a}, \qquad a = 1, \ldots, n \tag{90}$$

we can find a geometric interpretation of the integrability conditions

$$(\partial_\mu \Lambda_a)^2 = (\partial_\mu \Sigma_b)^2, \tag{91}$$

$$\partial_\mu \Lambda_a \partial^\mu \Sigma_b = 0, \qquad 1 \leqslant a, \qquad b \leqslant n. \tag{92}$$

Interestingly enough, in (2+1) dimensional spacetime, the standard integrable submodel constitutes the BPS sector of the CP^n model. That is, the solutions of the integrable subsystem (87), (88) are nothing else but the very well-known BPS solutions of the CP^n model [41–44]

$$U = \frac{\mathbf{f}}{|\mathbf{f}|}, \tag{93}$$

where $\mathbf{f}$ is a vector of $n+1$ (anti)holomorphic functions $f_i = f_i(z)$ of $z = x + \mathrm{i}y$, and $i = 0, \ldots, n$. Indeed, due to the conformal symmetry the general solution of the integrable submodel is just a collection of (anti)holomorphic functions

$$u_a = u_a(z) \quad \text{or} \quad u_a = u_a(\bar{z}). \tag{94}$$

Then, using the local $U(1)$ invariance one can express the BPS solutions of the CP^n model in the following form:

$$U = \frac{1}{\sqrt{1 + u_a^* u_a}} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix}. \tag{95}$$

Of course, as one usually wants to deal with objects carrying finite topological charge, the holomorphic function must be restricted to arbitrary rational functions.

In more than two spatial dimensions the conformal symmetry does not occur and the chance for the existence of an integrable submodel with a non-trivial set of solutions becomes limited. It is easy to see that a general solution of the integrability conditions is $u_a = A_a v$, where A_a are complex constants and v is any complex scalar field which satisfies the $(d + 1)$ dimensional scalar eikonal equation

$$\partial_\mu v \partial^\mu v = 0.$$

If one combines this result with the fact that such configurations should obey the dynamical equation (87) as well, then one may expect that the chances for non-trivial solutions of the integrable sector are highly reduced. This, of course, somehow restricts the range of applications of the obtained integrable sectors.

3.2. New integrable submodels

In this section we construct new integrable sectors in the CP^n model. In general, such alternative submodels are found if much less restrictive constraints are imposed.

We consider currents similar to (89)

$$J_\mu^{(a)} = \frac{\partial \mathcal{F}}{\partial u_a} \partial_\mu u_a - \frac{\partial \mathcal{F}}{\partial u_a^*} \partial_\mu u_a^*, \qquad a = 1, \ldots, n \tag{96}$$

but the function $\mathcal{F}$ is assumed to depend on the field variables in a specific manner. Namely,

$$\mathcal{F} = \mathcal{F}(u_1^* u_1 + \cdots + u_k^* u_k, u_{k+1}^* u_{k+1}, \ldots, u_n^* u_n). \tag{97}$$

Of course, we expect that the currents are conserved, i.e.,

$$\partial^\mu J_\mu^{(a)} = 0, \qquad a = 1, \ldots, n. \tag{98}$$

Thus, taking into account the field equations (86), we get the integrability conditions

$$\sum_{b=1}^{k} (u_a^* u_b^* \partial_\mu u_a \partial^\mu u_b - u_a u_b \partial_\mu u_a^* \partial^\mu u_b^*) = 0, \qquad a = 1, \ldots, n \tag{99}$$

$$\sum_{b=1}^{k} (u_a^* u_b \partial_\mu u_a \partial^\mu u_b^* - u_a u_b^* \partial_\mu u_a^* \partial^\mu u_b) = 0, \qquad a = 1, \ldots, n \tag{100}$$

$$u_a^* u_b^* \partial_\mu u_a \partial^\mu u_b - u_a u_b \partial_\mu u_a^* \partial^\mu u_b^* = 0, \qquad b = k + 1, \ldots, n \qquad a = 1, \ldots, n \tag{101}$$

$$u_a^* u_b \partial_\mu u_a \partial^\mu u_b^* - u_a u_b^* \partial_\mu u_a^* \partial^\mu u_b = 0, \qquad b = k + 1, \ldots, n \qquad a = 1, \ldots, n. \tag{102}$$

Once again, it is convenient to rewrite them using (90). Then,

$$\partial_\mu \Lambda_a \left(\sum_{b=1}^{k} \partial^\mu \Sigma_b \right) = 0, \qquad \partial_\mu \Sigma_a \left(\sum_{b=1}^{k} \partial^\mu \Lambda_b \right) = 0, \qquad a = 1, \ldots, k \tag{103}$$

$$\partial_\mu \Lambda_a \partial^\mu \Sigma_b = 0, \qquad \partial_\mu \Sigma_a \partial^\mu \Lambda_b = 0, \qquad a = 1, \ldots, n, \qquad b = k + 1, \ldots, n. \tag{104}$$

As we described in the Skyrme model in the previous section (or in the Faddeev–Niemi model in [18]), in an integrable sector the gradients of the pertinent real scalar fields (real coordinated on the target space) must be perpendicular. However, this does not mean that all gradients are mutually perpendicular. Varying the form of the $\mathcal{F}$ function we get that for some of the gradients it is sufficient to be perpendicular to a sum of others.

To obtain even weaker integrability conditions, and as a consequence a submodel with a larger set of solutions, one can take under consideration the following current:

$$J_\mu = \sum_{a=1}^{n} \left(\frac{\partial \mathcal{F}}{\partial u_a} \partial_\mu u_a - \frac{\partial \mathcal{F}}{\partial u_a^*} \partial_\mu u_a^* \right). \tag{105}$$

In this case the integrability conditions read

$$\left(\sum_{a=1}^{k} \partial_\mu \Lambda_a \right) \left(\sum_{b=1}^{k} \partial^\mu \Sigma_b \right) = 0, \tag{106}$$

$$\partial_\mu \Lambda_a \left(\sum_{b=1}^{k} \partial^\mu \Sigma_b \right) + \partial_\mu \Sigma_a \left(\sum_{b=1}^{k} \partial^\mu \Lambda_b \right) = 0, \qquad a = k+1, \ldots, n \tag{107}$$

$$\partial_\mu \Lambda_a \partial^\mu \Sigma_b + \partial_\mu \Sigma_a \partial^\mu \Lambda_b = 0, \qquad a = k+1, \ldots, n, \qquad b = k+1, \ldots, n. \tag{108}$$

To conclude, there is a family of integrable sectors of CP^n model. Integrability conditions vary from the strongest ones (the most restrictive)

$$\partial_\mu \Lambda_a \partial^\mu \Sigma_b = 0, \qquad a, b = 1, \ldots, n, \tag{109}$$

when $\mathcal{F} = \mathcal{F}(u_1 u_1^*, \ldots, u_n u_n^*)$ to the weakest ones (the least restrictive)

$$\left(\sum_{a=1}^{n} \partial_\mu \Lambda_a \right) \left(\sum_{b=1}^{n} \partial^\mu \Sigma_b \right) = 0, \tag{110}$$

if $\mathcal{F} = \mathcal{F}(u_1 u_1^* + \cdots + u_n u_n^*)$ and take all possible intermediate cases. Obviously, the full integrable submodel consists of the field equations (86) and a choice of an integrability condition, as discussed above.

It should be underlined that the construction of such integrable sectors of the CP^n model remains unchanged if we add a potential term to the Lagrangian [45].

Let us again interpret the derived integrable sectors in the context of CP^n model in $(2+1)$ dimensions. One can check that the most restrictive constraint (109) possesses, in addition to standard ones (94), the following solution:

$$\Lambda_a = \Lambda_a \left(\frac{z}{z^*} \right), \qquad \Sigma_a = \Sigma_a (z z^*), \tag{111}$$

where Λ_a, Σ_a are arbitrary functions depending on phase and modulus respectively. On the other hand it is known that the CP^n model also has non-BPS solutions, which can be constructed from the BPS solitons by acting with a projective operator [41–44]

$$U = \frac{P_+^k \mathbf{f}}{\left| P_+^k \mathbf{f} \right|}, \tag{112}$$

where

$$P_+ \mathbf{f} = \partial_z \mathbf{f} - (\mathbf{f}^\dagger \partial_z \mathbf{f}) \frac{\mathbf{f}}{|\mathbf{f}|^2}, \qquad k = 0, \ldots, n.$$

These non-BPS solutions (or at least some of them) obey the integrability conditions. Let us consider first the example $\mathbf{f} = (1, z, z^2)$, then for $k = 1$

$$U = \frac{1}{\sqrt{1 + 4|z|^2 + 6|z|^4 + 5|z|^6 + |z|^8}} \begin{pmatrix} z^*(1 + 2|z|^2) \\ |z|^4 - 1 \\ -z(2 + |z|^2) \end{pmatrix}. \tag{113}$$

Now, after expressing U by means of the parameterization (95) we get that the pertinent scalar complex functions are in the form (111). Since all non-BPS solutions by construction fulfil the dynamical equation of motions (86), we can conclude that new integrable sectors of the CP^n model in two spatial dimensions consist of BPS as well as non-BPS states.

It is not difficult to generalize this example to a whole class of non-BPS solutions which obey the same integrability condition (111). Indeed, let us choose for $\mathbf{f}$ $f_i = c_i z^{l_i}$, i.e., each f_i is a monomial $c_i z^{l_i}$ for integer l_i and arbitrary complex constant c_i. Further, we may assume $f_0 = 1$, i.e., $c_0 = 1$ and $l_0 = 0$ without loss of generality. Then it is easy to find that

$$P_+ f_i = c_i z^{l_i - 1} \left(l_i - \frac{\sum_{j=0}^n c_j^* c_j l_j \rho^{2l_j}}{\sum_{k=0}^n c_k^* c_k \rho^{2l_k}} \right) \equiv c_i z^{l_i - 1} g_i(\rho) \tag{114}$$

and

$$|P_+ \mathbf{f}|^2 = \frac{1}{\rho^2} \left(\sum_{j=0}^n c_j^* c_j l_j^2 \rho^{2l_j} - \frac{\left(\sum_{j=0}^n c_j^* c_j l_j \rho^{2l_j}\right)^2}{\sum_{k=0}^n c_k^* c_k \rho^{2l_k}} \right) \equiv \frac{K(\rho)^2}{\rho^2}, \tag{115}$$

where $z \equiv \rho \mathrm{e}^{\mathrm{i}\varphi}$. Therefore, for the components U_i of the vector U we find

$$U_i = \mathrm{e}^{-\mathrm{i}\varphi} c_i \, \mathrm{e}^{\mathrm{i} l_i \varphi} \frac{\rho^{l_i} g_i(\rho)}{K(\rho)} \equiv \mathrm{e}^{-\mathrm{i}\varphi} \frac{g_0(\rho)}{K(\rho)} \tilde{f}_i. \tag{116}$$

The overall phase factor $\mathrm{e}^{\mathrm{i}\varphi}$ can be skipped because of gauge invariance of the CP^n model. Further we have $\tilde{\mathbf{f}} = (1, \tilde{\mathbf{u}})$, i.e., $\tilde{f}_0 = 1$ and $\tilde{f}_i = \tilde{u}_a$ for $i = a = 1, \ldots, n$, and

$$\tilde{u}_a = c_a \, \mathrm{e}^{\mathrm{i} l_a \varphi} \frac{\rho^{l_a} g_a(\rho)}{g_0(\rho)}. \tag{117}$$

Obviously, $\tilde{\Lambda}_a = l_a \varphi$, whereas $\tilde{\Sigma}_a = \tilde{\Sigma}_a(\rho)$ is a function of ρ only, therefore the integrability condition (111) continues to hold.

As we see, in $(2+1)$ dimensions, there is a striking correspondence between integrability and BPS property of solutions. The strong integrable sector is just the BPS sector whereas weaker integrable sectors contain also non-BPS solutions. In higher dimensions we usually do not have any BPS solitons; however, the strong as well as weaker integrable sectors are still well defined. Thus, integrability is an important tool in investigating nonlinear models in higher dimensions.

An algebraic meaning of the family of integrable sectors can be easily found if one considers volume-preserving diffeomorphisms of the volume $2n$-form on the CP^n target space

$$\Omega = g(uu_1^*, \ldots, u_n u_n^*) \, \mathrm{d}u_1 \wedge \mathrm{d}u_1^* \wedge \cdots \wedge \mathrm{d}u_n \wedge \mathrm{d}u_n^*. \tag{118}$$

In other words we are interested in transformations $u_a \to v_a(u_1, u_1^*, \ldots, u_n, u_n^*)$ which leave the $2n$-form unchanged

$$\begin{aligned} \Omega &= g(uu_1^*, \ldots, u_n u_n^*) \, \mathrm{d}u_1 \wedge \mathrm{d}u_1^* \wedge \cdots \wedge \mathrm{d}u_n \wedge \mathrm{d}u_n^* \\ &= g(vv_1^*, \ldots, v_n v_n^*) \, \mathrm{d}u_1 \wedge \mathrm{d}v_1^* \wedge \cdots \wedge \mathrm{d}v_n \wedge \mathrm{d}v_n^*. \end{aligned} \tag{119}$$

Let us discuss infinitesimal transformations $u_a \to u_a + \epsilon_a$. Then

$$\mathrm{d}u_a \to \mathrm{d}u_a + \epsilon^a_{u_b} \, \mathrm{d}u_b + \epsilon^a_{u_b^*} \, \mathrm{d}u_b^*, \tag{120}$$

where summation over indices b is performed. Now, the invariance condition (119) gives

$$g\left(\epsilon^1_{u_1}+\epsilon^{*1}_{u_1^*}+\cdots+\epsilon^n_{u_n}+\epsilon^{*n}_{u_n^*}\right)+(\partial_1 g)(u_1\epsilon^{*1}+u_1^*\epsilon^1)+\cdots+(\partial_n g)(u_n\epsilon^{*n}+u_n^*\epsilon^n)=0. \quad (121)$$

Here $\epsilon^a_{u_a}\equiv\partial_{u_a}\epsilon^a$ and $\partial_1 g\equiv\partial_{u_1u_1^*}g$, etc. After introducing

$$\epsilon^a=g^{-1}F_{u_a^*} \quad (122)$$

we can rewrite equation (121) in a very compact form

$$\left(\partial_{u_1}\partial_{u_1^*}+\cdots+\partial_{u_n}\partial_{u_n^*}\right)(F+F^*)=0. \quad (123)$$

The general solution reads

$$F+F^*=\sum_i\zeta^i\left(u_1^{\sigma_i},\ldots,u_n^{\sigma_i}\right), \quad (124)$$

where $\sigma_i=1,2$ and $u^1\equiv u, u^2\equiv u^*$. However, from our point of view it is sufficient to investigate only pure imaginary solutions

$$F+F^*=0. \quad (125)$$

Thus, $F=\mathrm{i}G$ where $G=G(u_1,u_1^*,\ldots,u_n,u_n^*)$ is an arbitrary real function. It follows that all volume-preserving diffeomorphisms are generated by the vector fields

$$v^G=\mathrm{i}g^{-1}\sum_{a=1}^n\left(G_{u_a^*}\partial_{u_a}-G_{u_a}\partial_{u_a^*}\right), \quad (126)$$

which satisfy the Lie algebra

$$[v^{G_1},v^{G_2}]=v^{G_3}, \qquad G_3=\sum_{a=1}^n\left(G_{1u_a^*}G_{2u_a}-G_{1u_a}G_{2u_a^*}\right). \quad (127)$$

In order to find Abelian subalgebras we assume that $G_{1,2}$ take the following form:

$$G=G(u_1u_1^*+\cdots+u^ku_k^*,u_{k+1}u_{k+1}^*,\ldots,u_nu_n^*), \quad (128)$$

where $k=1,\ldots,n$ is a fixed number. Indeed, the corresponding commutator vanishes. Thus for each value of k we get an Abelian subalgebra $\mathcal{H}_k$. It is straightforward to see that these subalgebras form a tower of algebras

$$\mathcal{H}_1\supset\mathcal{H}_2\supset\cdots\supset\mathcal{H}_{n-1}. \quad (129)$$

Further, the vector fields v^G can be identified with the Noether charges of the currents (105), once the identification $G\to\mathcal{F}$ is made.

4. Conclusions

We have shown that the method, developed in [18] and originally devoted to the construction of integrable submodels for S^2 valued field theories, can be easily adapted for models with different target space, and defined in an arbitrary dimensional spacetime.

The interesting point is that the new integrable sectors of Skyrme as well as CP^n models derived here are defined by integrability conditions which are considerably weaker than the standard ones. Thus, one might expect that the set of solutions of the integrable subsystems is larger. In particular, it may cure the problem of the apparent non-existence of soliton solutions with a topological number larger than 1 in the standard integrable sector of the Skyrme model. However, the problem whether the new integrable submodel possesses such additional topological solutions is still an open question.

Moreover, a correspondence between BPS sector and strong integrability in the (2+1) dimensional CP^n model has been established. In addition, we have shown that weak integrable sectors support non-BPS solutions. Such a relation holds only in three-dimensional spacetime but it may indicate that in higher dimensions integrable submodels can play the role of BPS sectors. Therefore, one can hope that the construction of integrable submodels might compensate for the non-existence of BPS sectors and provide a useful tool for constructing solitons.

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